12-778 Fall 2023: Assignment #2 Solutions

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Solutions

Here I provide the solutions to this assignment. Some of the answers can be wrong, as I am far from infalible. So if you find an error, please report it to me!

Electrical Circuit Theory (15%)

Capacitance (5%):

For a capacitor with capacitance C and voltage V, why is the following equation true?

$$w = C \int_0^V v dv = \frac{1}{2} C V^2$$

> Answer
>
> This equation is describing the energy stored in the capacitor (w). It is easy to see that, because the charge stored by the capacitor is

Impedances (5%):

Is the resistor the only elementary circuit element that produces a voltage drop in a *dc* circuit? Why or why not?

> Answer
>
> The voltage across the three fundamental circuit elements are:
>
> - $v(t) = i(t)R$ for a resistor

- $v(t) = L\frac{di}{dt}$ for an inductor
- $v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)dt + v(t_0)$ for a capacitor

Each of these values *can* be different from zero depending on $i(t)$. If, for example, we assume that $i(t) =$ constant, then the inductor has a voltage of zero across it. Similarly, the capacitor would have zero voltage across it once completely discharged.

Frequency Response Functions (5%):

In class, we analyzed a simple circuit with an *ac* voltage source $v_s = v_i(t) = v_i e^{j\omega t}$ and two impedances in series, $Z_1 = R$ and $Z_2 = \frac{1}{j\omega C}$. We were interested in the voltage drop across Z_2 and by applying KVL found that $v_o(t) = \frac{1}{1+j\omega RC}v_i e^{j\omega t} = \frac{1-j\omega RC}{1+(\omega RC)^2}v_i e^{j\omega t}$. Knowing that we can easily find the magnitude and phase of this complex periodic signal $v_o(t)$ if we know its real (Re) and imaginary (Im) parts (i.e., $\sqrt{(Re)^2 + (Im)^2}$ and $\tan^{-1}\frac{(Im)}{(Re)}$) then what are they?

Answer

Multiplying the expression by $\frac{1-j\omega RC}{1-j\omega RC}$ we get things nicely sorted into:
$Re = \frac{1}{1+(\omega RC)^2}$
$Im = -\frac{\omega RC}{1+(\omega RC)^2}$

Signal Conditioning (30%)

Input signal range (10%)

In class we were trying to sample a signal that had a sinusoidal component with a frequency that was near 60Hz. Its instantaneous amplitude varied between 3V and -3V. When doing this we noticed two problems: (a) if we sampled at a low frequency (e.g., 1Hz) the resulting (aliased) signal was very low frequency and we could easily attribute this to the Nyquist-Shannon sampling theorem; and (b) the resulting signal would contain clusters of zero values repeating periodically. We had two hypothesis for the underlying cause of this second problem: saturation or inappropriately configured sampling on the ADC. After discarding the second hypothesis by using more accurate sampling methods, we were left with the saturation hypothesis. Upon further inspection of the resulting signals, we realized that the clusters of zero values would last for approximately 8 ms, repeat themselves every 16 ms, and occupy the part of the signal that would correspond to the negative values of the sine function. Thus, the saturation we were seeing was because the input signals to the ADC were lower than its minimum input voltage.

One can alleviate this problem by implementing a signal conditioning circuit that biases the signal to be entirely positive. A relatively simple circuit that can be used for this purpose would be a voltage divider. **Please provide a design for such a circuit including the values of the impedances (resistors in this case), an expression for the the source and output voltages, and proof that the signal would fall within the input range for the ADC in the Pico.**

> Answer
>
> Something like this:
> A bias voltage interface circuit

Interstage loading errors (10%)

When conneting different stages of our instrument (e.g., an analog device to an A/D converter), why is it that for current source inputs, we want to have Z_2 (the impedance of the next stage) be smaller than that of the source? Can you derive the expression for the loading error in this case (i.e., $e_I = I_2 - I_1$ where I_1 is the current that would circulate if we short-circuit the terminals of our first device – the source – and I_2 is the current that is pushed through the second device when connected)? Why is it $e_I = V_1 \frac{-Z_2}{Z_1^2 + Z_1 Z_2}$?

> Answer
>
> $$I_1 = \frac{V_1}{Z_1}$$
>
> $$I_2 = \frac{V_1}{Z_1 + Z_2}$$
>
> Thus, $e_I = I_2 - I_1 = V_1 \left(\frac{1}{Z_1 + Z_2} - \frac{1}{Z_1} \right)$
> And $e_I = V_1 \left(\frac{Z_1 - (Z_1 + Z_2)}{Z_1(Z_1 + Z_2)} \right)$
> Leading to $e_I = V_1 \frac{-Z_2}{Z_1^2 + Z_1 Z_2}$
> Which implies that for that error to be small, we want Z_2 to be smaller than Z_1.

Wheatstone Bridges (10%)

In class, we saw the derivation of the Wheatstone bridge and showed why the full-bridge had better properties for strain measurements than other configurations. We also said (but did not prove) that the half-bridge configuration was *better* than a voltage divider or the single/quarter bridge configuration. Here, I'm asking you to prove that.

Answer

Assuming that the four resistors of the bridge are R_1, R_2, R_3 and R_4, then in the quarter bridgecase, we assume $R_1 = R_3 = R_4$ and $R_2 = R_x$, and the expression for the output voltage V_o is:

$$V_o^{\text{quart}} = V\left(\frac{R_1}{R_x + R_1} - \frac{R_1}{R_1 + R_1}\right) = V\frac{R_1 - R_x}{2(R_x + R_1}$$

For the half bridge, we assume $R_1 = R_3$ and $R_2 = R_4 = R_x$, which leads us to:

$$V_o^{\text{half}} = V\left(\frac{R_1}{R_x + R_1} - \frac{R_x}{R_1 + R_x}\right) = V\frac{R_1 - R_x}{(R_x + R_1}$$

As we can see, V_o^{quart} is half as sensitive to changes in R_x as V_o^{half}.

Signal Characteristics (45%)

Solve problem 2.4 from Chapter 2 of Figliola (5%)

Answer

```
from numpy import mean, sqrt, square

y1 = [0, 11.76, 19.02, 19.02, 11.76, 0, -11.76, -19.02, -19.02, -11.76, 0]
y2 = [0, 15.29, 24.73, 24.73, 15.29, 0, -15.29, -24.73, -24.73, -15.29, 0]

y1_mean = mean(y1)
y2_mean = mean(y2)
y1_rms = sqrt(mean(square(y1)))
y2_rms = sqrt(mean(square(y2)))

print(f'The means are the same: y1_mean = {y1_mean}, y2_mean = {y2_mean} (a very small d
```

```
The means are the same: y1_mean = -4.844609562000683e-16, y2_mean = 3.2297397080004555e-16
```

Solve problem 2.7 from Chapter 2 of Figliola (5%)

Answer

As we know, a first order system has a natural (angular) frequencty $\omega = \sqrt{\frac{k}{m}}$ where m here is 1kg and k is 5000 N/cm, or 5×10^5 in units of kg/m^2. This leads to $\omega = 707.1$. Because this is the angular (or circular) frequency, its units are radians per second. To conver this into natural frequency f, we just divide by 2π to get $f = 112.5$ Hz.

Solve problem 2.11 from Chapter 2 of Figliola (10%)

Answer

We know that, in general, for a sum of a sine and a cosine with the same period, we can write: $B_n sin(\omega t) + A_n cos(\omega t) = C_n cos(\omega t - \phi)$, where $\phi = tan^{-1}\left(\frac{B_n}{A_n}\right)$ and $C_n = \sqrt{A_n^2 + B_n^2}$, then:

$$y(t) = \sum_{n=1}^{\infty}\left(\frac{2\pi n}{6} sin(n\pi t) + \frac{4\pi n}{6} cos(n\pi t)\right) = \sum_{n=1}^{\infty}\left(C_n cos(n\pi t - \phi)\right)$$

where $C_n = \sqrt{\left(\frac{2\pi n}{6}\right)^2 + \left(\frac{4\pi n}{6}\right)^2} = \frac{\sqrt{5}}{3}\pi n$ and $\phi = tan^{-1}\left(\frac{1}{2}\right)$
Or more succintly:

$$y(t) = \sum_{n=1}^{\infty}\frac{\sqrt{5}}{3}\pi n cos\left(n\pi t - tan^{-1}\left(\frac{1}{2}\right)\right)$$

Solve problem 2.15 from Chapter 2 of Figliola (15%)

Answer

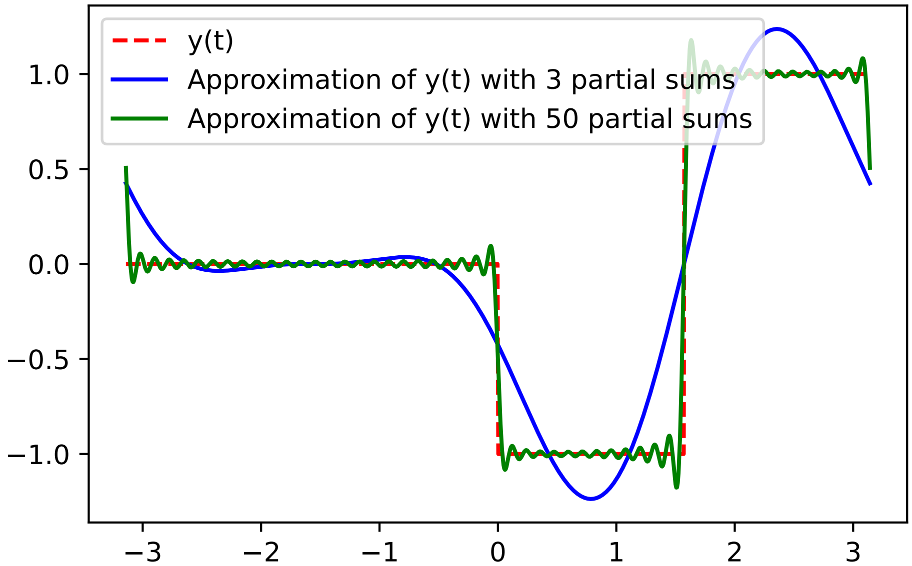
Using the formulas for A_0, A_n and B_n, and integrating the piece-wise function (composed of three parts), we get expressions. For the sake of brevity (and to save myself some time writing down LaTeX equations here, I'll just go straight to the answers:

$$A_0 = 0$$

$$A_n = \frac{-2}{\pi n} sin\left(\frac{\pi n}{2}\right)$$

$$B_n = \frac{2cos\left(\frac{n\pi}{2}\right) - cos(\pi n) - 1}{\pi n}$$

```
import numpy as np
from matplotlib import pyplot as plt

A0 = 0
A = lambda n: -2/(np.pi * n) * np.sin(np.pi*n/2)
B = lambda n: (2*np.cos(n/2*np.pi)-np.cos(np.pi * n) - 1)/(np.pi * n)

def y_hat(t,N):
  res = A0
  for n in range(1,N+1):
    res += A(n) * np.cos(n*t) + B(n) * np.sin(n*t)
  return res

t = np.linspace(-np.pi, np.pi, 1000)

y1 = np.zeros(500)
y2 = np.ones(250)*-1
y3 = np.ones(250)
y = np.concatenate((y1,y2,y3))

plt.plot(t,y, '--r',label='y(t)')
plt.plot(t,y_hat(t,3),'-b', label='Approximation of y(t) with 3 partial sums')
plt.plot(t,y_hat(t,50),'-g', label='Approximation of y(t) with 50 partial sums')
plt.legend()
plt.show()
```



Solve problem 2.24 from Chapter 2 of Figliola (10%)

Answer

There are many ways to answer this question, so I am just going to provide some generic guidance to the answers and avoid sketching the signal directly.

Output signal from a refrigerator's thermostat

The thermostat's output is a binary signal (i.e., *on* or *off*). Over time, the value is kept constant until the thermostat requires the compressor to be turned *on* or *off*, and that duration is proportional to the difference between the refrigerator's internal temperature and the setpoint temperature chosen by the user.

Electrical signal to a spark plug in a car engine

Every time the spark plug is needed (during ignition), there would be a very short burst (almost like an impulse) sent to it. So the signal would be a series of impulses spaced over time.

Input to a cruise control system

The cruise control signal would be similar to the thermostat one (i.e, it's the output of a controller) but not binary. Instead it would likely track the error between the speed of the car and the desired speed chosen by the user and vary over time depending on road conditions. A small apmlitude sinusodial signal with changing frequency could be used to show this...

Pure musical tone

A pure musical tone is composed of a single sine or cosine waveform at a specific frequency.

The note produced by a guitar string

Multiple harmonics of the fundamental frequency (the pure tone corresponding to the note) being played will be present in the resulting sound, but otherwise it would be similar to the previous one.

AM and FM radio signals

Amplitude modulation, as the name implies, is where the information that is being transmitted is encoded as a change in amplitude to a signal with constant frequency; whereas frequency modulation econdes information in changes to the frequency of a signal. Sinusoidal signals can be used here and modulated accordingly for each case.

High-speed sampling (10%)

Find an analog sensor that can be used to measure a physical phenomenon with interesting dynamic properties (e.g., accelerometer, microphone, etc.) and, using proper signal conditioning and sampling strategies, obtain measurements of this phenomenon using the sensor and your Raspberry Pi Pico at a sampling rate of at least 1000Hz. Compare the results of the sampling calling the ADC using `machine.ADC.read_u16()` and using the DMA trick we showed in class. Submit a sketch of the signal conditioning circuit, the Python code that needs to run on the Pico, two CSV files showing your sampled signals, and a written explanation of what you found during the process.

Answer

There is no one answer to this question.